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The googly amplitudes in gauge theory

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ABSTRACT: The googly amplitudes in gauge theory are computed by using the off shell MHV vertices with the newly proposed rules of Cachazo, Svrcek and Witten. The result is in agreement with the previously well-known results. In particular we also obtain a simple result for the all negative but one positive helicity amplitude when one of the external line is off shell.

KEYWORDS: Strong Coupling Expansion, QCD.

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1. Introduction

In a recent paper [1] the calculation of the perturbative amplitudes in gauge theory was reformulated by using the off shell MHV vertices. These MHV vertices are the usual tree level MHV scattering amplitudes in gauge theory [2, 3], continued off shell in a particular fashion as given in [1]. Some sample calculations were done in [1], sometimes with the help of symbolic manipulation. The correctness of the rules was partially verified by reproducing the known results for small number of gluons [4].

In view of the deep connection of the gauge theory with twistor space and string theory [5, 6], one would like to push the computation of the gauge theory amplitudes to its limit. By doing these calculations in the new formalism, we hope to learn more about the techniques of doing perturbative calculations in gauge theory or QCD [7, 8, 9], and to discover the inner structure of the amplitude as used in calculating loop amplitudes [9, 10]. A natural question is how to reproduce the exceptionally simple amplitudes with two positive helicity gluons and an arbitrary number of negative helicity ones, called googly amplitudes in [1]. These amplitudes were calculated from the string theory in [11]. As we will shown in this paper, the googly amplitudes can be calculated rather simply by following the new approach of [1]. By reproducing the previously well known results, our calculation gives a quite strong support to the Cachao-Svrcek-Witten proposal.

This paper is organized as follows. In section 2 we review briefly the spinor formalism and the rules of calculating the gauge theory amplitudes as proposed in [1]. For our purpose we will prove a general result for the all negative but one positive helicity amplitude when one of the external line is off shell. Two general formulas with spinors will also be proved. In section 3 we will compute the googly amplitude in the special case when the two positive helicity gluons are adjacent. By using an identity we reproduce the known result for this amplitude. The proof of the required identity is given in section 4. The calculation of the more general googly amplitudes is only briefly described. The detail for their calculations and the proof of a more general identity will appear in a separate publication.

2. The MHV vertices and some formulas with spinors

First let us recall the rules for calculating tree level gauge theory amplitudes as proposed in [1]. We will use the convention that all momenta are outgoing. By MHV we always mean an amplitude with precisely two gluons of negative helicity. If the two gluons of negative helicity are labeled as r, s (which may be any integers from 1 to n), the MHV vertices (or amplitudes) are given as follows:

$$V_n = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \,. \tag{2.1}$$

For an on shell (massless) gluon, the momentum in bispinor basis is given as:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} p_\mu = \lambda_a \tilde{\lambda}_{\dot{a}} \,. \tag{2.2}$$

For an off shell momentum, we can no longer define λ_a as above. The off-shell continuation given in [1] is to choose an arbitrary spinor $\tilde{\eta}^{\dot{a}}$ and then to define λ_a as follows:

$$\lambda_a = p_{a\dot{a}} \tilde{\eta}^{\dot{a}} \,. \tag{2.3}$$

For an on shell momentum p, we will use the notation λ_{pa} which is proportional to λ_a:

$$\lambda_{pa} \equiv p_{a\dot{a}} \tilde{\eta}^{\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \tilde{\eta}^{\dot{a}} \equiv \lambda_a \phi_p \,. \tag{2.4}$$

As demonstrated in [1], it is consistent to use the same $\tilde{\eta}$ for all the off shell lines (or momenta).

The 3 gluon vertex is very important for our purpose. As shown in figure 1, even if all the momenta are off shell, one can simplify this (MHV) vertex by using momentum conservation. We have

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \,, \tag{2.5}$$

$$V_3 = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{3} \langle \lambda_i, \lambda_{i+1} \rangle} = \langle \lambda_1, \lambda_2 \rangle = \langle \lambda_2, \lambda_3 \rangle = \langle \lambda_3, \lambda_1 \rangle \,. \tag{2.6}$$

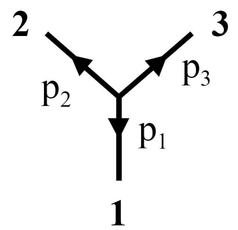

Figure 1: The 3 gluon vertex. It is not important which gluon has positive helicity.

By using only MHV vertices, one can build a tree diagram by connecting MHV vertices with propagators. For the propagator of momentum p, we assign a factor $1/p^2$. Any possible diagram (involving only MHV vertices) will contribute to the amplitude. As proved in [1], a tree level amplitude with q external gluons of negative helicity must be obtained from an MHV tree diagram with $q-1$ vertices. This counting can be generalized. As our purpose is to compute amplitudes with few external gluons with positive helicity, one easily prove that a tree level amplitude with n_+ external gluons of positive helicity will have no contribution from any diagram containing an MHV vertex with more than $n_+ + 2$ lines (not necessarily all internal). The reason is that there is not enough positive helicity lines because an internal line must have a positive helicity at one end and a negative helicity at the other

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end. Now we give a more precise formulation. If we assume that the number of the vertices with exactly i lines is n_i ($n_i \geq 0$) and let n_- denotes the external gluons with negative helicity, we have:

$$n_+ = \sum_i n_i(i-3) + 1\,, \tag{2.7}$$

$$n_- = \sum_i n_i + 1\,. \tag{2.8}$$

For $n_+ = 1$ we must have $n_i = 0$ for $i \geq 4$ and n_3 is given by $n_- - 1$ as derived [1]). For $n_+ = 2$ any contributing diagram will have exactly one MHV vertex with 4 lines, i.e. $n_4 = 1$. This paper will calculate only tree amplitude with 1 or 2 external gluons with positive helicity.

Our first result is a general expression for the off shell amplitude with all negative but one positive helicity:

$$V(1+, 2-, \ldots, n-) = \frac{p_1^2}{\phi_2\phi_n} \frac{1}{[2,3][3,4]\cdots[n-1,n]}\,, \tag{2.9}$$

where only the first particle with momentum p_1 is off-shell and has positive helicity. For $n=3$ this coincides with the off shell MHV vertices given in (2.1). Here we have used the following formula:

$$p_1^2 = (p_2+p_3)^2 = 2p_2\cdot p_3 = \langle 2,3\rangle[2,3]\,. \tag{2.10}$$

The other variant of (2.9) is for the case when the off shell gluon has negative helicity. We relabel this gluon to be the first gluon and the amplitude is given as follows:

$$V_n(1-, 2-, \ldots, r+, \ldots, n-) = \frac{\phi_r^4 p_1^2}{\phi_2\phi_n} \frac{1}{[2,3][3,4]\cdots[n-1,n]}\,, \tag{2.11}$$

where the r-th gluon has positive helicity. Now we prove the above results by the method of mathematical induction.

As we noted before, all contributing diagrams must contain only the 3 gluon vertex. By using this result one can decompose a multi-gluon amplitude into two sub-multi-gluon amplitude with less gluons, as shown in figure 2. As one can see from this figure, a multi-gluon amplitude with only one positive helicity can be decomposed into two multi-gluon amplitudes with less gluons. The two sub-multi-gluon amplitudes also have only one positive helicity and contain fewer gluons. In this way, we could prove eq. (2.9) by mathematical induction.

To begin the mathematical reduction, we first note that eq. (2.11) is true for $n=3$. We assume that it is also true for all $k \leq n$. We will prove that is is also true for $k = n+1$. In order to prove this, let us compute V_{n+1} from figure 2. We have

$$\begin{aligned} V_{n+1} = &\sum_{i=2}^{n} \left\{ \frac{p^2}{\phi_2\phi_i} \frac{1}{[2,3]\cdots[i-1,i]} \right\} \times \frac{1}{p^2} \times \frac{\langle\lambda_p,\lambda_q\rangle^3}{\langle\lambda_1,\lambda_p\rangle\langle\lambda_q,\lambda_1\rangle} \times \frac{1}{q^2} \times \\ &\times \frac{q^2}{\phi_{i+1}\phi_{n+1}} \frac{1}{[i+1,i+2]\cdots[n,n+1]}\,, \end{aligned} \tag{2.12}$$

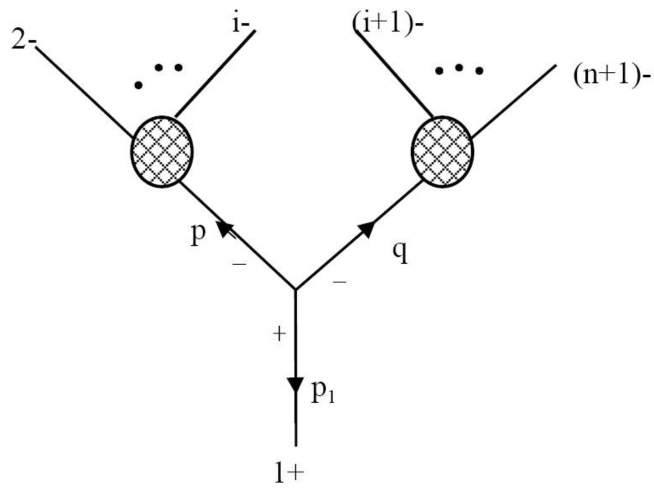

Figure 2: A multi-gluon amplitude with only one positive helicity can be decomposed into two multi-gluon amplitudes with less gulons. The two sub-multi-gluon amplitudes also have only one positive helicity. A summation over i should be understood.

by using the assumed result for all less multi-gluon amplitudes. Here

$$p = \sum_{l=2}^{i} p_l\,, \qquad \lambda_p = \sum_{l=2}^{i} \lambda_l \phi_l\,, \tag{2.13}$$

$$q = \sum_{l=i+1}^{n+1} p_l, \qquad \lambda_q = \sum_{l=i+1}^{n+1} \lambda_l \phi_l\,. \tag{2.14}$$

The degenerate cases for $i = 2$ and $i = n$ are also included correctly in the sum in (2.12) as one can easily verify. It is important to note here $\lambda_{p_2} = \lambda_2\phi_2$ which is not identical to λ_2.

By using eq. (2.6), we have

$$V_{n+1} = \frac{1}{\phi_2\phi_{n+1}} \frac{1}{[2,3]\cdots[n,n+1]} \sum_{i=2}^{n} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \langle\lambda_p,\lambda_q\rangle\,. \tag{2.15}$$

In order to simplify the above result we prove two formulas involving spinors. The first formula is:

$$\frac{[i,j]}{\phi_i\phi_j} = \frac{[i,l]}{\phi_i\phi_l} - \frac{[j,l]}{\phi_j\phi_l} = \frac{[i,l]}{\phi_i\phi_l} + \frac{[l,l]}{\phi_j\phi_l}\,, \tag{2.16}$$

where p_l is any on shell momentum (not necessarily be one of momenta in question). To prove the above formula one can assume $\tilde{\eta}^1 = 0$. (The general case is obtained by an SL(2) transformation.) We have

$$\begin{aligned}\frac{[i,j]}{\phi_i\phi_j} &= \frac{\tilde{\lambda}_{i1}\tilde{\lambda}_{j2} - \tilde{\lambda}_{i2}\tilde{\lambda}_{j1}}{\tilde{\lambda}_{i2}\tilde{\lambda}_{j2}(\tilde{\eta}_1)^2} = \frac{\tilde{\lambda}_{i1}}{\tilde{\lambda}_{i2}(\tilde{\eta}_1)^2} - \frac{\tilde{\lambda}_{j1}}{\tilde{\lambda}_{j2}(\tilde{\eta}_1)^2} \\ &= \left[\frac{\tilde{\lambda}_{i1}}{\tilde{\lambda}_{i2}(\tilde{\eta}_1)^2} - \frac{\tilde{\lambda}_{l1}}{\tilde{\lambda}_{l2}(\tilde{\eta}_1)^2}\right] + \left[\frac{\tilde{\lambda}_{l1}}{\tilde{\lambda}_{l2}(\tilde{\eta}_1)^2} - \frac{\tilde{\lambda}_{j1}}{\tilde{\lambda}_{j2}(\tilde{\eta}_1)^2}\right] \\ &= \frac{[i,l]}{\phi_i\phi_l} + \frac{[l,j]}{\phi_l\phi_j}\,.\end{aligned} \tag{2.17}$$

The other formula is:

$$B_n = \sum_{i=2}^{n-1} \frac{[i,i+1]}{\phi_i \phi_{i+1}} \langle \lambda_{p_2+\cdots+p_i}, \lambda_{p_{i+1}+\cdots+p_n} \rangle = (p_2 + p_3 + \cdots + p_n)^2 = p_1^2 \,. \tag{2.18}$$

where momenta conservation is used in the second equality. Now we prove this result. By using eq. (2.16), we have

$$\begin{aligned} B_n &= \sum_{i=1}^{n-1} \left[\frac{[i,l]}{\phi_i \phi_l} - \frac{[i+1,l]}{\phi_{i+1}\phi_l} \right] \langle \lambda_{p_2+\cdots+p_i}, \lambda_{p_{i+1}+\cdots+p_n} \rangle \\ &= \sum_{i=2}^{n} \frac{[i,k]}{\phi_i \phi_k} \langle \lambda_{p_i}, \lambda_{p_2+\cdots+p_n} \rangle \\ &= \frac{1}{\phi_k} \sum_{i=2}^{n} [i,k] \langle \lambda_i, \lambda_{p_2+\cdots+p_n} \rangle \\ &= \frac{1}{\phi_k} \sum_{i=2}^{n} \tilde\lambda_{i\dot a} \tilde\lambda_k^{\dot a} \lambda_{ia}\, \lambda^a_{p_2+\cdots+p_n} \\ &= (p_2+\cdots+p_n)_{a\dot a} (p_2+\cdots+p_n)^{\dot b a}\, \tilde\eta_{\dot b} \tilde\lambda_k^{\dot a} / \phi_k \\ &= -(p_2+\cdots+p_n)^2 \delta_{\dot a}^{\dot b} \tilde\eta_{\dot b}\, \tilde\lambda_k^{\dot a}/\phi_k = (p_2+\cdots+p_n)^2 \,. \end{aligned} \tag{2.19}$$

This ends the proof of eq. (2.18).

By using eq. (2.18) in (2.15) we have:

$$V_{n+1} = \frac{p_1^2}{\phi_2 \phi_{n+1}} \frac{1}{[2,3]\cdots[n,n+1]} \,, \tag{2.20}$$

as announced. This finishes the proof of (2.9) by mathematical induction. The other case when the off shell gluon has negative helicity is proved almost the same as above. We needn't care much about if the positive helicity particle is in the left blob or in the right blob in figure 2. The three gluon vertex doesn't differentiate much the ordering of the two different helicities as one can see from eq. (2.6).

3. The googly amplitude

Now we compute the n-particle googly amplitude. In this paper we concentrate on the easier case when the two positive helicity particles are adjacent. We label them to be 1 and n. As we noted in section 2, there is only one 4 line MHV vertex in any Feynman diagram. By using this result, the amplitude is computed by using the diagram decomposition as shown in figure 3.

All the 4 blob diagrams in figure 3 have been computed. By using the results proved in section 2, we have

$$\begin{aligned} A_n = &\frac{\phi_1^3 \phi_n^3}{[1,2][2,3]\cdots[n-1,n]} \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \frac{[i,i+1]}{\phi_i\phi_{i+1}} \frac{[j,j+1]}{\phi_j\phi_{j+1}} \frac{[k,k+1]}{\phi_k\phi_{k+1}} \times \\ &\times \frac{\langle V_2, V_3\rangle^3}{\langle V_1,V_2\rangle\langle V_3,V_4\rangle\langle V_4,V_1\rangle} \,, \end{aligned} \tag{3.1}$$

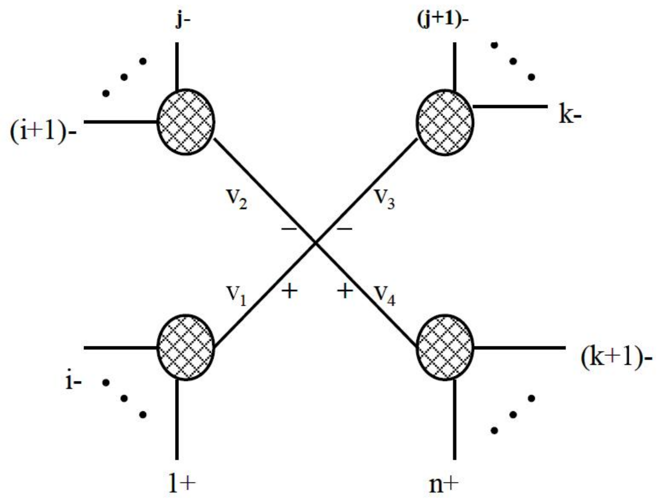

Figure 3: The decomposition of the googly amplitude $A_n(1+, 2-, \ldots, (n-1)-, n+)$. We note that there is only one 4 gluon vertex.

where

$$V_1 = \sum_{l=1}^{i} \lambda_l \phi_l \,, \qquad V_2 = \sum_{l=i+1}^{j} \lambda_l \phi_l \,, \tag{3.2}$$

$$V_3 = \sum_{l=j+1}^{k} \lambda_l \phi_l \,, \qquad V_4 = \sum_{l=k+1}^{n} \lambda_l \phi_l \,. \tag{3.3}$$

One can prove that the 3-fold summation in eq. (3.1) gives exactly the required result, i.e.

$$\sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \frac{[i,i+1]}{\phi_i \phi_{i+1}} \frac{[j,j+1]}{\phi_j \phi_{j+1}} \frac{[k,k+1]}{\phi_k \phi_{k+1}} \frac{\langle V_2, V_3 \rangle^3}{\langle V_1, V_2 \rangle \langle V_3, V_4 \rangle \langle V_4, V_1 \rangle} = \frac{[n,1]^3}{\phi_1^3 \phi_n^3} \,. \tag{3.4}$$

Eq. (3.4) was not proved by doing the summation directly. In fact we prove it by analyzing its pole terms and prove that all the pole terms are vanishing. The finite value can be obtained by choosing a special configuration of ϕ_i and doing the summation directly. We will do this in the next section.

By using eq. (3.4) in eq. (3.1), we have

$$A_n(1+, 2-, \ldots, (n-1)-, n+) = \frac{[n,1]^3}{\prod_{i=1}^{n-1} [i, i+1]} \,. \tag{3.5}$$

This is the known result for the googly amplitude [2, 3]. It is the complex conjugate of the MHV amplitude, eq. (2.1) for Minkowski signature.

For the generic googly amplitude, the needed diagram decomposition is given in figure 4. Depending on whether the positive helicity gluon is in the second blob or in the

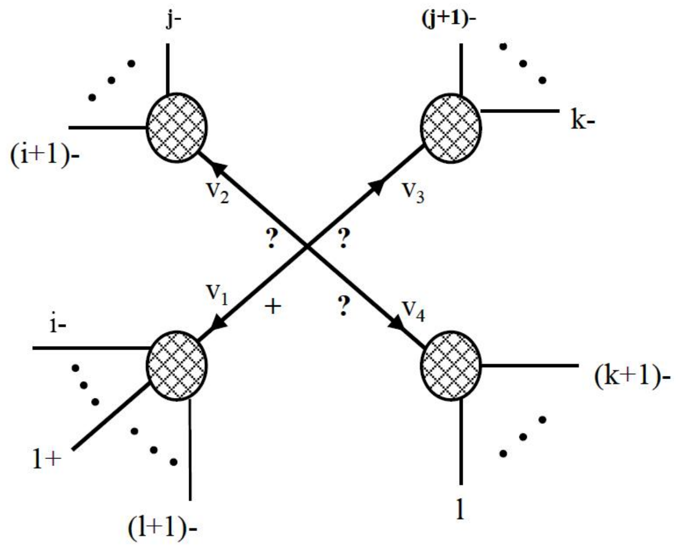

Figure 4: The decomposition for the generic googly amplitude. We note that the helicity of the internal line can be different, depending where we put the positive helicity gluon.

third blob or in the last blob, the expression for the 4 line vertex should be changed appropriately. The 4-fold summation should give the generic googly amplitude by using a more complicated identity. The required identity is also proved by a similar analysis as given in the next section for the proof of eq. (3.4). The details will be given in a separate publication.

4. The proof of eq. (3.4)

In this section we will prove eq. (3.4). As all the quantities are written in an SL(2) invariant form, one can choose a convenient $\tilde{\eta}$ to simplify the writing of the expressions. We will choose $\tilde{\eta}^1 = 0$ and $\tilde{\eta}^2 = 1$. Then we have

$$\phi_i = \tilde{\lambda}_{i2}\,, \tag{4.1}$$

and

$$\frac{[i,j]}{\phi_i\phi_j} = \frac{\tilde{\lambda}_{i1}}{\tilde{\lambda}_{i2}} - \frac{\tilde{\lambda}_{j1}}{\tilde{\lambda}_{j2}}\,. \tag{4.2}$$

If we do a rescaling of $\tilde{\lambda}_{i1}$ by $\tilde{\lambda}_{i2}$, i.e. by defining $\varphi_i = \tilde{\lambda}_{i1}/\tilde{\lambda}_{i2}$, and also do a rescaling of λ_{ia} by $1/\tilde{\lambda}_{i2}$, then eq. (3.4) becomes:

$$\begin{aligned} F(\varphi) = \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1} &(\varphi_i - \varphi_{i+1})(\varphi_j - \varphi_{j+1})(\varphi_k - \varphi_{k+1}) \times \\ &\times \frac{\langle V_2, V_3\rangle^3}{\langle V_1, V_2\rangle\langle V_3, V_4\rangle\langle V_4, V_1\rangle} = (\varphi_n - \varphi_1)^3\,, \end{aligned} \tag{4.3}$$

where

$$V_1 = \sum_{l=1}^{i} \lambda_l\,, \qquad V_2 = \sum_{l=i+1}^{j} \lambda_l, \tag{4.4}$$

$$V_3 = \sum_{l=j+1}^{k} \lambda_l\,, \qquad V_4 = \sum_{l=k+1}^{n} \lambda_l\,. \tag{4.5}$$

There are also two constraints:

$$V_1 + V_2 + V_3 + V_4 = \sum_{l=1}^{n} \lambda_l = 0\,, \tag{4.6}$$

$$\sum_{l=1}^{n} \lambda_i\, \varphi_i = 0\,, \tag{4.7}$$

from momentum conservation.

If we assume that λ_1 and λ_n are solved explicitly in terms of the rest λ_i and all φ_j (containing only linear terms in φ_j for $2 \le j \le n-1$), then the left hand side of eq. (4.3) can be considered as a function of λ_j for $2 \le j \le n-1$ and all φ_j. As a function of φ we will show that it is independent of φ_j for $2 \le j \le n-1$.

First one easily shows that there is no pole terms for $\varphi_j \to \infty$ for $2 \le j \le n-1$. This is because there is at most a φ_j^2 in the numerator and the denominator factor $\langle V_1, V_2\rangle\langle V_3, V_4\rangle\langle V_4, V_1\rangle$ grows as φ_j^2 at least because V_1 and V_4 contain linear terms in φ_j and V_2 and V_3 don't depend on φ_j. There is no φ_j^2 term in $\langle V_4, V_1\rangle$ although V_1 and V_4 both contain a $\varphi_j\lambda_j$ term.

It remains to show all the finite pole terms are vanishing. The possible pole terms appear if any factor of $\langle V_1, V_2\rangle\langle V_3, V_4\rangle\langle V_4, V_1\rangle$ is vanishing. Let us consider first the vanishing of $\langle V_1, V_2\rangle$. We denote this set of V_1 and V_2 as v_1 and v_2:

$$v_1 = \lambda_1 + \cdots + \lambda_{n_1}\,, \qquad v_2 = \lambda_{n_1+1} + \cdots + \lambda_{n_2}\,. \tag{4.8}$$

As one can see from figure 5 that there are contributions from summing over k and fixing $i = n_1$ and $j = n_2$. The residues (ignoring an overall factor $(\varphi_{n_1} - \varphi_{n_1+1})(\varphi_{n_2} - \varphi_{n_2+1})$) for these pole terms are:

$$C_1 = \sum_{k=n_2+1}^{n-1} (\varphi_k - \varphi_{k+1})\, \frac{\langle v_2, V_3\rangle^3}{\langle v_2, V_3\rangle\langle V_4, v_1\rangle}\,. \tag{4.9}$$

Because $\langle v_1, v_2\rangle = 0$ we have $v_1 = cv_2$. By using this result and the relation $v_1 + v_2 + V_3 + V_4 = 0$ in the above we have

$$\begin{aligned} C_1 &= \frac{1}{c(c+1)} \sum_{k=n_2+1}^{n-1} (\varphi_k - \varphi_{k+1})\, \langle v_2, V_3\rangle \\ &= \frac{1}{c(c+1)} \left[\sum_{k=n_2+1}^{n-1} \varphi_k \langle v_2, \lambda_k\rangle - \phi_n \langle v_2, \lambda_{n_2+1} + \cdots + \lambda_{n-1}\rangle \right] \\ &= \frac{1}{c(c+1)} \sum_{k=n_2+1}^{n} \langle v_2, \varphi_k \lambda_k\rangle\,. \end{aligned} \tag{4.10}$$

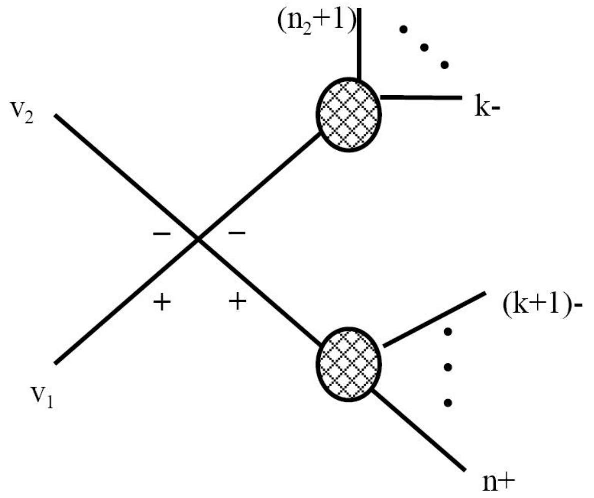

Figure 5: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_1, V_2\rangle$. There is a summation over k.

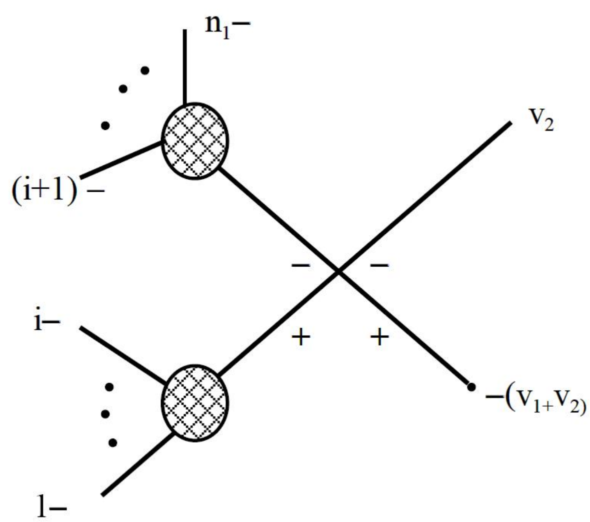

Figure 6: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_3, V_4\rangle$. There is a summation over i.

Similar pole terms can also be obtained from the vanishing of the factor $\langle V_3, V_4\rangle$ in figure 6 by setting $V_3 = v_2$ and $V_4 = -(v_1 + v_2)$ and summing over i. This give a contribution:

$$\begin{aligned} C_2 &= \sum_{i=1}^{n_1-1} (\varphi_i - \varphi_{i+1}) \frac{\langle V_2, v_2\rangle^3}{\langle V_1, V_2\rangle\langle -(v_1+v_2), V_1\rangle} \\ &= \frac{1}{c(c+1)} \sum_{i=1}^{n_1-1} (\varphi_i - \varphi_{i+1}) \langle v_2, V_1\rangle \\ &= \frac{1}{c(c+1)} \sum_{i=1}^{n_1} \langle v_2, \varphi_i \lambda_i\rangle \,. \end{aligned} \tag{4.11}$$

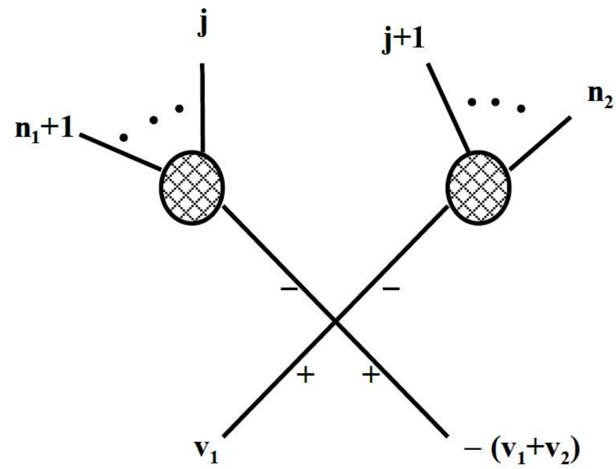

Figure 7: The pole terms $\langle v_1, v_2\rangle$ from the factor $\langle V_4, V_1\rangle$. There is a summation over j.

The last piece of the pole terms is from the vanishing of the factor $\langle V_4, V_1\rangle$ in figure 7 by setting $V_1 = v_1$ and $V_4 = -(v_1 + v_2)$ and summing over j. The result is:

$$C_3 = \frac{1}{c(c+1)} \sum_{j=n_1+1}^{n_2} \langle v_2, \varphi_j \lambda_j\rangle \, . \tag{4.12}$$

By summing the three residues together, we have

$$C_1 + C_2 + C_3 = \frac{1}{c(c+1)} \sum_{i=1}^{n} \langle v_2, \varphi_i \lambda_i\rangle = 0 \, , \tag{4.13}$$

by using the constraint eq. (4.7). This proves that there is no pole terms for the function $F(\phi)$. So $F(\phi)$ must be independent of all φ_j for $2 \leq j \leq n-1$.

Having proved the independence of $F(\varphi)$ on φ_j, let us compute this function explicitly. As it is independent of φ_j for $2 \leq j \leq n-1$, we can choose a special set of φ. A convenient choice is as follows:

$$\varphi_i = x \, , \qquad \text{and} \qquad \varphi_3 = \cdots = \varphi_{n-1} = y \, . \tag{4.14}$$

For generic x and y (and generic λ_i which we don't say explicitly), one can show that all the possible $\langle V_1, V_2\rangle$, $\langle V_3, V_4\rangle$ and $\langle V_4, V_1\rangle$ for different choices of i, j and k are non-vanishing. For our choice of φ, the possible non-vanishing factor for $(\varphi_i - \varphi_{i+1})(\varphi_j - \varphi_{j+1})(\varphi_k - \varphi_{k+1})$ is from $i = 1$, $j = 2$ and $k = n-1$ only. We have then

$$F(\varphi) = (\varphi_1 - x)(x - y)(y - \varphi_n) \, \frac{\langle \lambda_2, \lambda_3 + \cdots + \lambda_{n-1}\rangle^3}{\langle \lambda_1, \lambda_2\rangle \langle \lambda_3 + \cdots + \lambda_{n-1}, \lambda_n\rangle \langle \lambda_n, \lambda_1\rangle} \, . \tag{4.15}$$

By using the two constraints eq. (4.6) and eq. (4.7), we have

$$\lambda_1 = \frac{(x - \varphi_n)\lambda + (y - \varphi_n)\mu}{\varphi_n - \varphi_1} \, , \qquad \lambda_n = -\frac{(x - \varphi_1)\lambda + (y - \varphi_1)\mu}{\varphi_n - \varphi_1} \, , \tag{4.16}$$

by setting $\lambda_2 = \lambda$ and $\lambda_3 + \cdots + \lambda_{n+1} = \mu$. By using these results in eq. (4.16), we obtained exactly $F(\varphi) = (\varphi_n - \varphi_1)^3$. This completes the proof of eq. (3.4).

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References

[1] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, hep-th/0403047.

[2] S.J. Parke and T.R. Taylor, *An amplitude for N gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[3] F.A. Berends and W.T. Giele, *Recursive calculations for processes with N gluons*, *Nucl. Phys.* **B 306** (1988) 759.

[4] M.L. Mangano and S. Parke, *Multiparton amplitudes in guage theories*, *Phys. Rept.* **200** (1991) 30.

[5] R. Penrose, *Twistor algebra*, *J. Math. Phys.* **8** (1967) 345; *Thee central programme of twistor theory*, *Chaos, Solitons, and Fractal* **10** (1999) 581.

[6] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, hep-th/0312171.

[7] Z. Bern, *String based perturbative methods for gauge theories*, hep-ph/9304249.

[8] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[9] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109 [hep-ph/9602280].

[10] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040]; Z. Bern, A. De Freitas and L.J. Dixon, *Two-loop helicity amplitudes for quark gluon scattering in QCD and gluino gluon scattering in supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **06** (2003) 028 [hep-ph/0304168].

[11] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, *J. High Energy Phys.* **04** (2004) 012 [hep-th/0402016]; R. Roiban and A. Volovich, *All googly amplitudes from the B-model in twistor space*, hep-th/0402121.